

# Maths Semester II

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## 1 First Order Differential Equations(ODEs)

### 1.1 Separable ODEs

A separable ODE is one in which the functions of both  $x$  and  $y$  can be separated.

$$\frac{dy}{dx} = \frac{g(x)}{f(y)}$$
$$\frac{dy}{dx} f(y) = g(x)$$

This can then be rearranged and integrated so that

$$\int f(y)dy = \int g(x)dx$$

Example

$$\frac{dy}{dx} = e^{x+y} = e^x e^y$$
$$\int e^{-y} dy = \int e^x dx$$
$$-e^{-y} = e^x + c$$
$$e^x + e^{-y} = c$$

$c$  is just a constant so its sign does not matter at this point.  
We can then solve explicitly for  $y$ .

$$e^{-y} = c - e^x$$
$$y = -\ln(c - e^x)$$

## 1.2 Boundary and Initial Conditions

The general solution to a 1st Order DE always contains one undefined constant of integration, like  $c$ . A boundary (for  $x$ ) or initial (for  $t$ ) condition is given, typically  $y(x)$  at a given  $x$  value.

Example:  $T$  goes from  $90^\circ\text{c}$  to  $70^\circ\text{c}$  in 10 minutes( $t$ ), the room temperature is  $20^\circ\text{c}(T_0)$ , find  $T$  after 20 minutes.

$$\begin{aligned}\frac{dT}{dt} &= -\alpha(T - T_0) \\ \int \frac{dT}{T - T_0} &= -\alpha \int dt \\ \ln(T - T_0) &= -\alpha t + c \\ T - T_0 &= e^{-\alpha t + c} = Ae^{-\alpha t} \\ A &= e^c\end{aligned}$$

We can now use our initial conditions to solve for  $T(20)$

When  $t = 0$

$$T - T_0 = 90 - 20 = 70$$

$$A = 70$$

When  $t = 10$

$$T = T_0 + Ae^{-\alpha t}$$

$$70 = 20 + 70e^{-10\alpha}$$

$$e^{-10\alpha} = \frac{70 - 20}{70} = \frac{5}{7}$$

When  $t = 20$

$$T = 20 + 70e^{-20\alpha}$$

$$(e^{-10\alpha})^2 = e^{-20\alpha}$$

$$T = 20 + 70 \left(\frac{5}{7}\right)^2 = 55.7^\circ\text{c}$$

### 1.3 Homogeneous ODEs

Replace  $y$  with  $yt$  and  $x$  with  $xt$ , if all the  $t$ s cancel then the ODE is homogeneous.

$$\begin{aligned}\frac{dy}{dx} &= f\left(\frac{y}{x}\right) \\ \text{let } v &= \frac{y}{x} \text{ so } y = vx \\ \frac{dy}{dx} &= \frac{d}{dx}vx\end{aligned}$$

$v$  is a function of  $x$  so we must use the product rule.

$$\begin{aligned}\frac{dy}{dx} &= v + x\frac{dv}{dx} = f(v) \\ \frac{dv}{dx} &= \frac{f(v) - v}{x}\end{aligned}$$

This is now a separable function, and can be solved as before.

Example:  $y(1) = 1$

$$\begin{aligned}\frac{dy}{dx} &= \frac{y}{x^2}(x - y) = \frac{y}{x} - \left(\frac{y}{x}\right)^2 \\ \text{let } v &= \frac{y}{x} \text{ so } y = vx \\ \frac{dy}{dx} &= v + x\frac{dv}{dx} = v - v^2 \quad \text{separable} \\ \int \frac{-dv}{v^2} &= \int \frac{dx}{x} \\ \frac{1}{v} &= \ln x + c = \frac{x}{y} \\ y &= \frac{x}{\ln x + c} \\ y(1) &= 1 \\ 1 &= \frac{1}{\ln(1) + c} \\ c &= 1\end{aligned}$$

## 1.4 Linear ODEs

Linear ODEs always take the form of

$$\frac{dy}{dx} + y \cdot P(x) = Q(x)$$

When  $Q(x) = 0$  this becomes a separable ODE and can be solved as before such that

$$\begin{aligned} dy \cdot \frac{1}{y} &= -P(x)dx \\ y &= Ae^{\int -P(x)dx} \quad \text{where } A = e^c \end{aligned}$$

When  $P(x) = 0$  this becomes a simple separable ODE and can be solved by integration.

Recall the product rule  $\frac{d}{dx}(u \cdot v) = u'v + uv'$   
This is strikingly close to our Linear ODE, except we are missing a factor, the integration factor,  $\mu(x)$ . We want  $\mu$  to be such that  $\mu' = P(x)\mu$ , allow us to do some algebraic manipulation.

$$\begin{aligned} \mu' &= P(x)\mu \\ P(x) &= \frac{\mu'}{\mu} = \frac{d}{dx} \ln(\mu) \\ \int P(x)dx &= \ln(\mu) \\ \mu &= e^{\int P(x)dx} \end{aligned}$$

We can then multiply by our integrating factor, then apply the chain rule, however this is best shown with an example.

$$\begin{aligned} \cos(x) \frac{dy}{dx} + \sin(x)y &= 1 \\ \frac{dy}{dx} + \tan(x)y &= \sec(x) \quad \text{Linear ODE} \\ \mu &= e^{\int \tan(x)dx} \\ \int \tan(x)dx &= \ln(\sec(x)) \\ \mu &= \sec(x) \quad \text{Multiply ODE by } \mu \\ \frac{dy}{dx} \sec(x) + \sec(x) \tan(x)y &= \sec^2(x) \end{aligned}$$

Since  $\frac{d}{dx} \sec(x) = \sec(x) \tan(x)$  we should now be able to see how the LHS of our equation can be withdrawn into its chain rule form.

$$\begin{aligned} \frac{d}{dx} \sec(x) \cdot y &= \frac{dy}{dx} \sec(x) + \sec(x) \tan(x)y \\ \frac{d}{dx} \sec(x) \cdot y &= \sec^2(x) \\ \sec(x) \cdot y &= \int \sec^2(x) = \tan(x) + c \\ y &= \sin(x) + c \cdot \cos(x) \end{aligned}$$

## 1.5 Exact ODEs

When a variable in a multi variable function  $f(x, y)$  has variables that rely on another e.g.  $y = y(t), x = x(t)$  then the derivative of that function is known as the total derivative and is given as

$$\frac{d}{dt} f(x, y) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

This can also occur for when one variable is the function of another e.g.  $x = x, y = y(x)$ , the total derivative of such is

$$\frac{d}{dx} f(x, y) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} \quad (1)$$

Exact ODEs take the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0$$

Where M and N must be separated by a plus. As you can see, this is in the same form as a the derivative of the function from equation (1), thus we can assume that

$$\frac{\partial f}{\partial x} = M \quad \frac{\partial f}{\partial y} = N$$

In order to verify this assumption we can test an 'exactness condition'

$$\begin{aligned} \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial M}{\partial y} &= \frac{\partial N}{\partial x} \end{aligned}$$

We must now find a function to satisfy these conditions, however this is best left to an example.

$$\begin{aligned}
x + y^2 + 2xy \frac{dy}{dx} &= 0 \\
M = x + y^2 \quad N &= 2xy \\
\frac{\partial M}{\partial y} &= 2y \\
\frac{\partial N}{\partial x} &= 2y
\end{aligned}$$

This satisfies the 'exactness condition', and so we can begin to find the function that solves the ODE.

$$\begin{aligned}
M &= \frac{\partial f}{\partial x} \\
\int M dx = f &= \frac{1}{2}x^2 + xy^2 + g(y) \\
N &= \frac{\partial f}{\partial y} \\
\int N dy = f &= xy^2 + h(x)
\end{aligned}$$

For  $\int M dx = \int N dy$  to be true their constants must satisfy each other, thus  $g(y) = 0, h(x) = \frac{1}{2}x^2$

$$f(x, y) = \frac{1}{2}x^2 + xy^2 + c = 0$$

## 2 Second Order ODEs

### 2.1 Introduction

They take the form of

$$m(x)\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$

$r(x) = 0$  Homogeneous  
 $r(x) \neq 0$  Inhomogeneous

The general solution is

$$y(x) = Ay_1(x) + By_2(x)$$

Where  $y_1$  &  $y_2$  are two functions of  $x$  and  $A$  &  $B$  are integration constants. We normally assume we have constant coefficients, as in driven oscillators.

$$m\frac{d^2y}{dx^2} + b\frac{dy}{dx} + ky = F_0 \cos(\omega t)$$

### 2.2 Homogeneous

Consider the second order homogeneous ODE

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

We normally take an educated guess at a solution, e.g.  $y = e^{mx}$ , where  $m$  is some constant. Thus

$$\frac{dy}{dx} = me^{mx} \quad \frac{d^2y}{dx^2} = m^2e^{mx}$$

Subbing this back into our original homogeneous equation and then dividing by  $e^{mx}$  gives us

$$(m^2 + am + b) = 0 \quad \text{auxillary equation}$$
$$(m - m_1)(m - m_2) = 0 \quad m_1 \text{ \& } m_2 \text{ are roots}$$

There are three cases to consider, when  $m_1$  &  $m_2$  are

- a. distinct and real
- b. distance and complex
- c. the same

### 2.2.1 Case a.

Where  $m_1 \neq m_2$  and both  $m_1$  &  $m_2$  are real. From our educated guess we know

$$y_1(x) = e^{m_1 x} \quad y_2(x) = e^{m_2 x}$$

So the general solution is

$$y(x) = Ae^{m_1 x} + Be^{m_2 x}$$

Example:

$$\begin{aligned} 3\frac{d^2 y}{dx^2} + -5\frac{dy}{dx} + -2y &= 0 \\ 3m^2 - 5m - 2 &= 0 \quad \text{auxillary equation} \\ (3m + 1)(m - 2) &= 0 \\ m_1 &= \frac{-1}{3} \\ m_2 &= 2 \\ y(x) &= Ae^{-x/3} + Be^{2x} \quad \text{general solution} \end{aligned}$$

### 2.2.2 Case b.

Where  $m_1 \neq m_2$  and both  $m_1$  &  $m_2$  are complex. We know that

$$y_1(x) = e^{m_1 x} \quad y_2(x) = e^{m_2 x}$$

But now  $y_1$  &  $y_2$  are complex. This means we can rewrite the general solution using Euler's Formula.

$$\begin{aligned} \text{let } m_1 &= \mu + i\lambda \\ \text{let } m_2 &= \mu - i\lambda \end{aligned}$$

So the general solution is

$$\begin{aligned} y(x) &= Ae^{m_1 x} + Be^{m_2 x} = Ae^{(\mu+i\lambda)x} + Be^{(\mu-i\lambda)x} \\ &= e^{\mu x} [Ae^{i\lambda x} + Be^{-i\lambda x}] \\ &= e^{\mu x} [A(\cos(\lambda x) + i \sin(\lambda x)) + B(\cos(\lambda x) - i \sin(\lambda x))] \\ &= e^{\mu x} [(A + B)(\cos(\lambda x)) + (A - B)i \sin(\lambda x)] \\ &= e^{\mu x} [(C)(\cos(\lambda x)) + (D) \sin(\lambda x)] \end{aligned}$$

The final general solution is

$$e^{\mu x} [(C)(\cos(\lambda x)) + (D) \sin(\lambda x)]$$

Where  $C = A + B$  &  $D = i(A - B)$  are both integration constants.



### 2.2.3 Case c.

Where  $m_1 = m_2$ . Thus

$$y_1 = e^{m_1 x}$$

So we need another solution. We will prove this in a later section.

$$y_2(x) = xe^{m_1 x}$$

$$y(x) = Ay_1(x) + By_2(x) \quad \text{general solution}$$

$$y(x) = (A + Bx)e^{m_1 x}$$

Example:

$$\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

$$m^2 + 2m + 1 = (m + 1)^2 = 0 \quad \text{auxillary equation}$$

$$m = -1$$

$$y_1(x) = e^{-x}$$

$$y_2(x) = xe^{-x}$$

$$y(x) = (A + Bx)e^{-x}$$

general solution

We can check that  $y_2$  is a solution.

$$\frac{dy_2}{dx} = e^{-x} - xe^{-x} = (1 - x)e^{-x}$$

$$\frac{d^2 y_2}{dx^2} = -e^{-x} - e^{-x} + xe^{-x} = (x - 2)e^{-x}$$

$$\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

original ODE

$$[(x - 2) + 2(1 - x) + x]e^{-x} = 0$$

$$0 = 0$$